Second semester 2012-2013
Midsemestral exam
Algebraic Number Theory
M.Math.IInd year
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Q 1. Define a fractional ideal I in an integral domain A. If I is a fractional ideal such that there exists a fractional ideal J with $IJ = A$, then prove $J = \{x \in K : xI \subset A\}$.

OR

If A is a Dedekind domain, prove that the group $\mathcal{P}$ of principal fractional ideals is isomorphic to K^*/A^* where K is the quotient field of A and A^* denotes the subgroup of units in A.

Q 2. If L/K be a Galois extension of algebraic number fields, and P is a non-zero prime ideal of O_K, prove that the Galois group permutes the prime ideals lying over P transitively.

OR

Let A be an integrally closed domain. Let $f \in A[X]$ be a monic polynomial so that $f = gh$ where $g, h \in K[X]$ are monic with K, the quotient field of A. Prove that $g, h \in A[X]$.

Q 3. Let K be an algebraic number field. If p is a prime number and $pO_K = P_1^{e_1} P_2^{e_2} \cdots P_g^{e_g}$ where P_i are prime ideals of O_K, prove that the set of prime ideals of O_K lying over p is precisely $\{P_1, \cdots, P_g\}$.

OR

If $\alpha \in O_K$ for an algebraic number field K of degree n, then prove that the norm of the ideal αO_K is $|N_{K/\mathbf{Q}}(\alpha)|$.

Q 4. If $d \equiv 2$ or 3 mod 4, is a square-free positive integer, and $b > 0$ is the smallest positive integer such that $db^2 \pm 1$ is a perfect square, (say a^2), prove that $a + b\sqrt{d}$ is the fundamental unit of $\mathbf{Q}(\sqrt{d}$.

Q 5. If a prime number q splits into an even number of prime ideals in $\mathbf{Q}(\zeta_p)$, show that it must split into two primes in $\mathbf{Q}(\sqrt{(-1)^{(p-1)/2}p})$.

OR

Consider $K = \mathbf{Q}(\zeta_n)$, where ζ_n is a primitive n-th root of unity. Show that a prime $p \in \mathbf{Z}$ splits completely in $\mathcal{O}_K$ if, and only if, $p \equiv 1 \bmod n$.

Q 6. Let K be a cubic extension of $\mathbf{Q}$ such that $-49 \leq \text{disc }(K) < 0$. Use the Minkowski bound to deduce that K has class number 1.
Hint: Use the fact that the sign of the discriminant of a number field with s complex places is $(-1)^s$.

OR

Consider a number field K with r real, and $2s$ non-real embeddings into $\mathbf{C}$ over $\mathbf{Q}$. Write down a map which maps $\mathcal{O}_K$ onto a lattice in $\mathbf{R}^n$ where $n = r + 2s$. Further, prove that the volume of a fundamental parallelotope for this lattice is $\frac{\sqrt{|d_K|}}{2^s}$.

Q 7. If L/K is a Galois extension of number fields, and Q is a prime ideal of O_L lying over and unramified over a prime ideal P of O_K, define the Frobenius automorphism $Frob(Q/P) = \left[\frac{L/K}{Q}\right]$. If $L \supset E \supset K$, find the relation between $Frob(Q/P)$ and $Frob(Q/Q \cap O_E)$.

Q 8. Prove that $x^2 + 5 = y^3$ does not have any integer solutions.
Hint: Note (as in question 6) that 3 does not divide the class number of $\mathbf{Q}(\sqrt{-5})$.

Q 9. If K is a complete field with respect to a nonarchimedean valuation $|.|_K$ and L is a finite extension of K, obtain the unique extension of $|.|_K$ to L which gives a nonarchimedean valuation on it.